Taylor expansion of homogeneous functions on a page

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Let $U \subseteq \mathbb{R}^n$ be open with $\lambda x \in U$ for all $x \in U$ and $\lambda > 0$. Let $f : U \to \mathbb{R}$ be differentiable and *homogeneous*, i.e.

$$f(\lambda x) = \lambda f(x) \text{ for all } x \in U \text{ and } \lambda > 0. \tag{1}$$

Euler's Theorem for homogeneous function states that

$$\langle \nabla f(x), x \rangle = f(x) \text{ for all } x \in U, \tag{2}$$

as can be seen by computing

$$\langle \nabla f(\lambda x), x \rangle = \frac{d}{d\lambda} f(\lambda x) = \frac{d}{d\lambda} \lambda f(x) = f(x). \tag{3}$$

By Euler's Theorem, the Taylor polynomial of first order of f reads

$$T^{(1)} f(x; y-x) = f(x) + \langle \nabla f(x), y-x \rangle = \langle \nabla f(x), y \rangle . \tag{4}$$

The constant term vanishes and $T^{(1)} f$ is a homogeneous polynomial in y.

Now let $g : U \to \mathbb{R}$ be continuously differentiable and *homogeneous of degree* 2, i.e.

$$g(\lambda x) = \lambda^2 g(x) \text{ for all } x \in U \text{ and } \lambda > 0. \tag{5}$$

Euler's Theorem implies

$$\frac{1}{2} \langle \nabla g(x), x \rangle = g(x) \text{ and } H_g(x)\, x = \nabla g(x), \tag{6}$$

where H_g denotes the Hessian of g. Hence, the Taylor polynomial of second order of g reads

$$\begin{aligned} T^{(2)} g(x; y-x) &= g(x) + \langle \nabla g(x), y-x \rangle + \frac{1}{2}(y-x)^{\mathrm{T}} H_g(x)(y-x) \\ &= \frac{1}{2} x^{\mathrm{T}} H_g(x)\, x + x^{\mathrm{T}} H_g(x)\, y - x^{\mathrm{T}} H_g(x)\, x + \frac{1}{2} y^{\mathrm{T}} H_g(x)\, y \\ &\quad - x^{\mathrm{T}} H_g(x)\, y + \frac{1}{2} x^{\mathrm{T}} H_g(x)\, x = \frac{1}{2} y^{\mathrm{T}} H_g(x)\, y. \end{aligned} \tag{7}$$

The constant and linear terms vanish and $T^{(2)} g$ is a homogeneous polynomial of degree 2 in y. If again f is homogeneous of degree 1, then

$$f(y) \sim \sqrt{T^{(2)} f^2(x; y-x)} = \sqrt{y^{\mathrm{T}} \left[\frac{1}{2} H_{f^2}(x) \right] y}. \tag{8}$$

In economic applications, f may be a model that aggregates risk measurements x_k to an overall risk $f(x)$. In this context, matrix [. . .] is called *sensitivity-implied tail-correlation matrix*.

Further reading

Joachim Paulusch and Sebastian Schlütter: Sensitivity-implied tail-correlation matrices. *Journal of Banking and Finance 134(106333)* 2022

Joachim Paulusch and Sebastian Schlütter: Taylor expansion of homogeneous functions. arXiv:2107.06526v4 2024