Nested Euler Allocation on a page

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Let $U \subseteq \mathbb{R}^n$ be open with $\lambda x \in U$ for all $x \in U$ and $\lambda > 0$. Let $f : U \to \mathbb{R}$ be differentiable and *homogeneous*, i.e.

$$f(\lambda x) = \lambda f(x) \text{ for all } x \in U \text{ and } \lambda > 0. \tag{1}$$

Euler's Theorem for homogeneous function states that

$$\langle \nabla f(x), x \rangle = f(x) \text{ for all } x \in U, \tag{2}$$

as can be seen by computing

$$\langle \nabla f(\lambda x), x \rangle = \frac{d}{d\lambda} f(\lambda x) = \frac{d}{d\lambda} \lambda f(x) = f(x). \tag{3}$$

Note that $\langle \nabla f(x), x/\|x\| \rangle$ is the directional derivative of f in direction $x/\|x\|$. We call the partial derivatives

$$\omega_k = \omega_k(f, x) = \partial_{x_k} f(x) \tag{4}$$

sensitivites. An economic interpretation is as follows: The coordinates of x are regarded as risk measurements with respect to different sources of risk, and $f(x)$ is the aggregated risk. If a risk x_k changes by Δx_k, the aggregated risk changes up to first order by $\omega_k \Delta x_k$. Euler's Theorem implies

$$\omega_1 x_1 + \ldots + \omega_n x_n = f(x). \tag{5}$$

So, $\omega_k x_k$ may be seen as the risk contribution of risk k to the aggregated risk $f(x)$. This way of re-allocation of the aggregated risk to risk sources is named *Euler allocation*.

Suppose that the ℓ-th coordinate y_ℓ of f is modeled by some differentiable and homogeneous function $g(x)$, while the other coordinates of f are fixed. Then, the chain rule implies

$$\omega_k(f \circ g, x) = \omega_\ell(f, g(x))\, \omega_k(g, x). \tag{6}$$

We mention the special case of *square root formulas*. Let $A \in \mathbb{R}^{n \times n}$ and $x^{\mathrm{T}} A x > 0$ for all $x \in U$. The function

$$f(x) = \sqrt{x^{\mathrm{T}} A x} \tag{7}$$

is homogeneous and we may apply Euler allocation. The sensitivities of f are given by

$$\omega(f, x) = \frac{Ax}{\sqrt{x^{\mathrm{T}} A x}} = \frac{Ax}{f(x)}. \tag{8}$$

Further reading

Joachim Paulusch: The solvency II Standard Formula, Linear Geometry, and Diversification. *Journal of Risk and Financial Management 10(11)* 2017