Local portfolio optimization on a page

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We denote a *portfolio* by

$$p = \begin{pmatrix} p_1 \\ \vdots \\ p_N \end{pmatrix} \in \mathbb{R}^N.$$

Risk of the portfolio is given by a *risk function* $p \mapsto f(p) \in \mathbb{R}$, two times continuously differentiable in a neighborhood of p, e.g. with real, symmetric, positive definite matrix $S = (\sigma_{ij})_{i,j=1}^N$ (covariance matrix)

$$f(p) = \sigma(p) = \sqrt{p^{\mathrm{T}} S p} = \left\{ \sum_{i,j=1}^{N} p_i p_j \sigma_{i,j} \right\}^{1/2}$$

$\sigma(p)$ then is the *volatility* of portfolio p. – Expected *return* is given by $p \mapsto \mu(p) \in \mathbb{R}$. Being an expected value, return is linear, i.e. with $\mu \in \mathbb{R}^N$

$$\mu(p) = \mu_1 p_1 + \ldots + \mu_N p_N = \mu^{\mathrm{T}} p = \langle \mu, p \rangle$$

By the Theorem of Taylor we have in a neighborhood of p

$$f(p + v) = f(p) + \langle \nabla f(p), v \rangle + \mathcal{O}(\|v\|^2),$$

with

$$\nabla f(p) = \partial f(p)^{\mathrm{T}} = (\partial_1 f(p), \ldots, \partial_N f(p))^{\mathrm{T}} \in \mathbb{R}^N$$

Theorem 1. *Let $c > 0$ and*

$$\mu^* = \mu - \frac{\nabla f(p)}{\|\nabla f(p)\|^2} \langle \nabla f(p), \mu \rangle \neq 0.$$

Then there is a unique optimized portfolio p^* with $\|p^* - p\| = c$, $\langle \nabla f(p), p^* - p \rangle = 0$, and

$$\langle \mu, p + v \rangle < \langle \mu, p^* \rangle \text{ for all } v \in \mathbb{R}^N,\ \|v\| = c,\ \langle \nabla f(p), v \rangle = 0,\ v \neq p^* - p,$$

namely

$$p^* = p + c \frac{\mu^*}{\|\mu^*\|}.$$

Proof. It holds that $\|p^* - p\| = c$ and

$$\langle \nabla f(p), p^* - p \rangle = \left\langle \nabla f(p), c \frac{\mu^*}{\|\mu^*\|} \right\rangle = \frac{c}{\|\mu^*\|} \left(1 - \frac{\langle \nabla f(p), \nabla f(p) \rangle}{\|\nabla f(p)\|^2} \right) \langle \nabla f(p), \mu \rangle = 0.$$

Note that $\langle \mu, v \rangle = \langle \mu^*, v \rangle$ for all $v \in \mathbb{R}^N$ with $\langle \nabla f(p), v \rangle = 0$. Therefore

$$\langle \mu, p + v \rangle = \langle \mu, p \rangle + \langle \mu^*, v \rangle = \langle \mu, p \rangle + c \|\mu^*\| \cos \alpha$$

for all $v \in \mathbb{R}^N$, $\|v\| = c$, $\langle \nabla f(p), v \rangle = 0$, wherein α denotes the angle between μ^* and v. $\cos \alpha = 1$ if and only if $v = p^* - p$. □