Noether's Theorem on a page

In the following we give a proof of Noether's Theorem. The proof is taken from [1], Section 20. We consider a Lagrangian system

$$L : TM \longrightarrow \mathbb{R},$$

where TM is the tangent space of a differentiable Manifold M and L is a smooth function, the Lagrangian. We say that the system has a *symmetry* if there exists a one-parameter group of diffeomorphisms

$$h^s : M \longrightarrow M, \qquad (s \in \mathbb{R})$$

with the property

$$L(Th^s(q, \dot{q})) \equiv L(h^s(q), T_q h^s(\dot{q})) = L(q, \dot{q}) \text{ for all } (q, \dot{q}) \in TM,$$

where the first equivalence just serves to clarify the notation and T_q denotes the derivative of h^s at q. A *first integral* of the Lagrangian system is a smooth function

$$I : TM \longrightarrow \mathbb{R},$$

which is constant along any solution of the Lagrangian system, and the existence of such a first integral is called a *conservation law.* We prove

Noether's Theorem. *To every symmetry of a Lagrangian system there corresponds a conservation law.*

Proof. Let φ be a solution and $(q, \dot{q})$ be local coordinates of TM. Then φ fulfills the Euler-Lagrange equations

$$\frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}.$$

Set

$$\Phi(s, t) = h^s(\varphi(t)),$$

and denote d/ds with a prime, d/dt with a dot. Then

$$0 = \frac{d}{ds}L(\Phi, \dot{\Phi}) = \frac{\partial L}{\partial q}\Phi' + \frac{\partial L}{\partial \dot{q}}\dot{\Phi}' = \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)\Phi' + \frac{\partial L}{\partial \dot{q}}\left(\frac{d}{dt}\Phi'\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\Phi'\right).$$

This means that

$$I(q, \dot{q}) = \frac{\partial L}{\partial \dot{q}}\, \frac{d}{ds}h^s(q)\bigg|_{s=0}$$

is a first integral of the Lagrangian system. "In fact, I is the rate of change of $L(x, v)$ when the vector $v \in TM_x$ varies inside TM_x with velocity $(d/ds)|_{s=0}\, h^s(x)$." ([1], p. 89)

References

[1] V. I. Arnold: *Mathematical Methods of Classical Mechanics.* Graduate Texts in Mathematics 60, second edition. Springer (1989)